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AN EXAMINATION OF THE EFFECTS OF THE CRITERION FUNCTIONAL ON OPTIMAL FIRE-SUPPORT POLICIES

by

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1. Introduction.

As one of the authors has pointed out in [30], for the purposes of military operations research, it is convenient to consider that there are three essential parts of any time-sequential combat optimization problem:

(a) the decision criteria (for both combatants),

(b) the model of conflict termination (and/or unit breakpoints),

(c) the model of combat dynamics.

An important problem of military operations research is the determination of the relationship between the nature of system objectives and the structure of optimal (time-sequential) combat strategies. Of particular importance is the sensitivity of the structure of optimal combat strategies to the nature of military objectives.† In a time-sequential combat optimization problem the combatant objectives are quantified through the criterion functional (see [4]). If the optimal combat strategy and associated payoff are quite sensitive to the functional form of the criterion functional, then care must be exercised in the selection of the functional form.

An important constituent part of fire support is the target allocation function which matches a specific weapon type with an acquired target within the target's environment.†† It is not surprising then that the determination of optimal target allocation strategies for supporting weapon systems††† is (in one form or another) one of the most extensively studied problems in both the open literature (see [33] (or [34]) for further references) and also classified sources. During World War II the problem of the appropriate mixture of tactical and strategic air forces (another aspect of the optimal fire-support strategy problem) was extensively debated by experts. Some analysis details are to be found in the classic book by Morse and Kimball (see pp. 73-77

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†See [23] for a discussion of the influences of political objectives on military objectives for the evaluation of (time-sequential) combat strategies.

††See pp. I-33 to I-43 of [19] for a discussion of the key elements of the fire-support system for purposes of systems analysis.

†††See [38] for a brief discussion of the distinction between a "primary" weapon system (or infantry) and a "supporting" weapon system.

of [21]). This problem was further studied at RAND in the late 1940's and early 1950's (see [7]) and elsewhere (see [1]). It would probably not be too far-fetched to say that this problem stimulated early research on both dynamic programming (see [2]) and also differential games (see [7], [12]). Today the problem of the determination of optimal air-war strategies (another aspect of the fire-support problem) is being rather extensively studied by a number of organizations (see, for example, [8], [16], [36]).

Thus, the objective of this investigation is to determine the sensitivity of the optimal time-sequential fire-support policy to the functional form of the criterion functional. Our research approach is to combine Lanchester-type models of warfare (see, for example, [28]-[30] and references contained therein) with generalized control theory† (i.e. optimization theory for dynamic systems). This general research program has been described in more detail elsewhere [31], [32]. It seems appropriate to examine sensitivity of the optimal policy by considering a concrete problem. Consequently, our research approach is to consider several different criterion functionals for the same tactical situation involving a time-sequential allocation of supporting fires. The tactical situation that we have chosen to examine is the "approach to contact" during an assault on enemy defensive positions by friendly ground forces. We seek to determine the "best" allocation for the supporting fires of the friendly forces. We will consider a mathematically tractable version†† of this problem so that we can make quantitative comparisons among the optimal policies corresponding to the various criterion functionals. Corresponding to each different criterion functional is a different optimization (here optimal control) problem. Each of these problems has been solved, and the corresponding optimal fire-support policies will be contrasted.

In this paper four different criterion functionals are considered: it is shown that both the difference and the ratio of military worths of friendly and enemy survivors

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†This term was apparently first coined by Y. C. Ho in [9] (see also [10]).

††Weiss [38] has emphasized that a simplified model of a combat situation is particularly valuable when it leads to a clearer understanding of significant relationships which would tend to be obscured in a more complex model.

(computed according to linear utilities) and also the ratio of the military worths of friendly and enemy losses as criterion functionals may lead to exactly the same optimal policy. A completely different optimal policy, however, is obtained for the weighted average of force ratios of opposing infantry (at the time that the supporting fires are lifted) as the criterion functional. We have decided that the three former criterion functionals (i.e. the difference and the ratio of the military worths of survivors and the ratio of the military worths of losses) are appropriate for an "attrition" objective,[†] whereas the weighted average of force ratios is appropriate for a "breakthrough" objective.[††] [In the latter case, the attacking force tries to overpower the defenders at one place along a front and then pour reinforcements through the break in the defender's defenses in order to "penetrate" behind the enemy lines and, for example, disrupt enemy command, control, and communications.]

The body of this paper is organized in the following fashion. First, we review previous work on the relationship between the quantification of military objectives and the structure of optimal time-sequential fire-distribution policies in order to place the work at hand in proper perspective. Then we describe the fire-support problem and discuss the four criterion functionals that will be used to determine optimal fire-support policies. Each of these criterion functionals represents a different quantification of military objectives, and all appear to be reasonable criteria. Next, the optimal time-sequential fire-support policies are described for the four problems. The structures of the four optimal policies are then contrasted. Next, we justify the optimization results that we have been discussing by sketching their development via modern optimal control theory. This development is given for each of the four problems. Finally, we discuss what we have learned from our investigation of the dependence of the structure of optimal time-sequential fire-support policies on the quantification of military objectives.

[†]In other words, the friendly forces seek an "overall" military advantage.

[††]In other words, the friendly forces seek a "local" military advantage.

2. Previous Work on the Structure of Optimal Fire-Distribution Policies.

The only systematic examinations of the influences of the nature of the criterion function on the structure† of optimal time-sequential fire-distribution strategies known to the authors are those of Taylor [24]-[27], [31], [34], [35]. In [24]-[27] and [31] a linear utility†† was assumed for the military worth of the number of each surviving weapon system type, and the criterion functional (payoff) was taken to be the net military worth of survivors (i.e. the difference between the military worths of friendly and enemy forces). Taylor (see [24]-[27] and [31]) has studied how the optimal time-sequential fire-distribution policy depends on the assignment of these linear utilities. In other words, he examined the sensitivity of the optimal time-sequential combat policy to parametric variations in the assigned linear utilities for survivors. It has been shown that the n-versus one fire-distribution problems studied in [24]-[27] all have quite simple solutions when enemy survivors are valued in direct proportion to their kill capabilities (as measured by their Lanchester attrition-rate coefficients (see [28]-[29]) against the (homogeneous) friendly forces).

Pugh and Mayberry [23] have suggested††† that an appropriate payoff, or objective function (in our terminology, criterion functional), for the quantitative evaluation of combat strategies is the loss ratio (calculated possibly using weighting factors for heterogeneous forces). They have stated [23] that an "almost equivalent" criterion is the loss difference. In this paper we will examine to what extent these criteria

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†In [25] and [31] the influences of the nature of the target-type attrition process on the structure of optimal time-sequential fire-distribution policies are examined.

††See [11] for methodology for the development of these linear utilities. For optimal control/differential game combat optimization problems, the assumption of linear utilities yields that the boundary conditions for the adjoint variables (at least when no terminal state constraint is active) are independent of the values of the state variables. Serious computational difficulties may arise when nonlinear utilities are assumed. The effects of assuming nonlinear utilities for military resources upon the evaluation of time-sequential combat strategies has apparently never been studied.

†††However, Pugh and Mayberry [23] do not explore the consequences of various functional forms for the criterion functional.

are in fact equivalent. In combat problems with either no replacements or a fixed-length planning horizon, it is readily seen that minimizing the loss difference is the same as maximizing the difference in survivors. It is such a case of no replacements that we will examine here. It remains to determine the "equivalence" of minimizing the loss ratio to maximizing the ratio of survivors and to relate these results to those for maximizing the difference in survivors.

Furthermore, for the evaluation of combat strategies it is of interest to consider the military worth (i.e. utility of military resources) of survivors. In almost all† the work that has appeared in the open literature†† a linear utility has been assumed for valuation of survivors, and some form of net military worth (i.e. the difference between the military worths of friendly and enemy survivors) has been taken as the payoff (i.e. criterion functional) (see, for example, [20], [24]-[27], [31]-[32], [35]). One reason for assuming such linear utilities is that of mathematical tractability: the boundary conditions for the dual variables do not depend on the state variable values (at least when no terminal constraint involving the state variables is active).

The only study known to the authors of the consequences on nonlinear utilities for survivors is contained in [34], where Kawara's supporting weapon system game [14] is examined. Taylor [34] has determined (at least for the case in which the appropriate side's (in Kawara's case, the defender) supporting weapon system is not annihilated) the most general form of the criterion functional which leads to optimal fire-support strategies being <u>independent</u> of force levels, and he has shown that the criterion functional chosen by Kawara [14] is a special case of this form. In other words, Taylor has shown that Kawara's conclusion [14] that optimal fire-support strategies do not

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†The only exceptions known to the authors are the papers by Chattopadhyay [5] and Kawara [14]. For example, in Kawara's paper [14] the payoff is the ratio of opposing infantry strengths (measured in terms of total numbers) at the "end of battle" (see also the differential game studied in Appendix D of [34]).

††A comprehensive review of pertinent literature published prior to 1973 in the field of optimizing time-sequential tactical decisions (using Lanchester-type models of warfare) is to be found in [32].

depend on force levels only applies to problems with the special type of criterion functional used by Kawara and is not true in general. No other examination of the dependence of optimal combat strategies on combatant objectives is known to the authors.

3. Comparison of Optimal Fire-Support Policies.

In this section we give the fire-support allocation problem for which the optimal policy is developed according to four different criterion functionals. These time-sequential fire-support policies are then compared.

3.1. The Fire-Support Problem.

Let us consider the attack of heterogeneous X forces against the static defense of heterogeneous Y forces along a "front." Each side is composed of primary units (or infantry) and fire-support units (or artillery). The X infantry (denoted as X_1 and X_2) launches an attack against the positions held by the Y infantry (denoted as Y_1 and Y_2). We may consider X_1 and X_2 to be infantry units operating on spatially separated pieces of terrain. We assume that the X_1 infantry unit attacks the Y_1 infantry unit and similarly for X_2 and Y_2 with no "crossfire" (e.g. the X_1 infantry is not attrited by the Y_2 infantry). We will consider only the "approach to contact" phase of the battle. This is the time from the initiation of the advance of the X_1 and X_2 forces towards the Y_1 and Y_2 defensive positions until the X_1 and X_2 forces actually make contact with the enemy infantry in "hand-to-hand" combat. It is assumed that this time is fixed and known to X.

The X_i forces begin their advance against the Y_i forces from a distance and move towards the Y_i position. The objective of the X_i forces during the "approach to contact" is to close with the enemy position as rapidly as possible. Accordingly, small arms fire by the X_i forces is held at a minimum or firing is done "on the move" to facilitate rapid movement. It is not unreasonable, therefore, to assume that the effectiveness of X_i force "on the move" is negligible against Y_i.[†] We assume, however,

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[†]It may be shown that such an approximation is necessary for reasons of mathematical tractability in the fire-support optimal control problem to be subsequently given.

that the defensive Y_i fire (for $i = 1,2$) causes attrition to the advancing X_i forces in their "field of fire" at a rate proportional to only the number of Y_i firers. Let a_i denote the constant of proportionality. It is convenient to refer to the attrition of a target type as being a "square-law" process when the casualty rate is proportional to the number of enemy firers only and as being a "linear-law" process when it is proportional to the product of the numbers of enemy firers and remaining targets (see [25]-[27]). Brackney [3] has shown that a "square-law" attrition process occurs† when the time to acquire targets is negligible in comparison with the time to destroy them. He has pointed out that such a situation is to be expected to occur when one force assaults another. Additionally, we assume that either the Y forces have no fire-support units or their fire support is "organic" to the Y units (i.e. fire-support units are integrated with Y_i and only those with Y_i support Y_i.

During the "approach to contact" the X fire-support units (denoted as W) deliver "area fire" against the Y_i forces.†† Let ϕ_i denote the fraction of the W fire-support units which fire at Y_i. [We then have that $\phi_1 + \phi_2 = 1$ and $\phi_i \geq 0$ for $i = 1,2$.] Then for constant ϕ_i there are a constant number of fire-support units firing at Y_i, since we assume that the W fire-support units are not in the combat zone and do not suffer attrition. In this case, the Y_i attrition rate is proportional to the Y_i force level (see [37]; also [13]). Let c_i denote the corresponding constant of proportionality. This combat situation is shown diagrammatically in Figure 1.

It is the objective of the X forces to utilize their fire-support units (denoted as W) over time in such a manner so as to achieve the "most favorable" situation at the end of the "approach to contact" at which time the force separations between

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†To be precise, one can only conjecture that such an attrition process probably occurs under the stated conditions.

††In other words, we assume that X's fire-support units fire into the (constant) area containing the enemy's infantry without feedback as to the destructiveness of this fire.

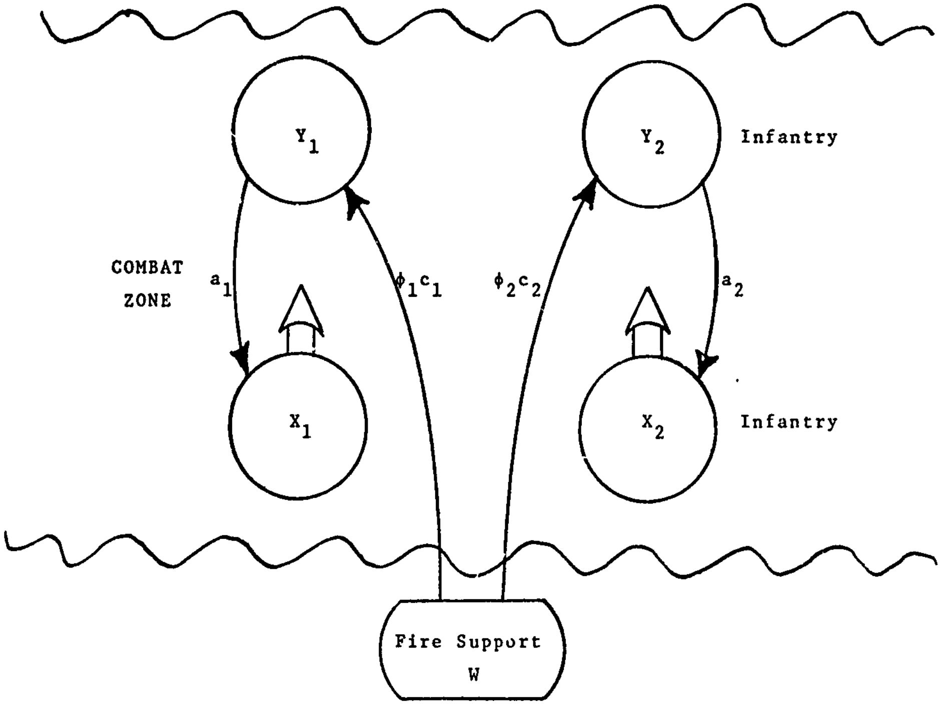

Figure 1. Diagram of Fire-Support Problem Considered for Examination of Effect of Criterion Functional on Optimal Fire-Support Policy.

opposing infantries are zero and artillery fires must be lifted from the enemy's positions in order not to also kill friendly forces. The "outcome" of this phase of battle may be measured in several different ways and is quantitatively expressed through the criterion functional (denoted as J). Thus, we have the following optimal control problem for the determination of the optimal time-sequential fire-support allocation policy (denoted as $\phi^*(t)$ for $0 \leq t \leq T$, where T denotes the time of the end of the "approach to contact") for the W fire-support units.

$$\underset{\phi_i(t)}{\text{maximize}}\ J,$$

with stopping rule: $t_f - T = 0$,

subject to: (battle dynamics)
$$\frac{dx_i}{dt} = -a_i y_i, \qquad (1)$$
$$\frac{dy_i}{dt} = -\phi_i c_i y_i \quad \text{for } i = 1,2,$$

with initial conditions:

$$x_i(t=0) = x_i^o \quad \text{and} \quad y_i(t=0) = y_i^o \quad \text{for } i = 1,2,$$

and

$x_1, x_2, y_1, y_2 \geq 0$ (State Variable Inequality Constraints),

$\phi_1 + \phi_2 = 1$ and $\phi_i \geq 0$ for $i = 1,2$ (Control Variable Inequality Constraints),

where

J denotes the criterion functional,

$x_i(t)$ denotes the number of X_i infantry at time t, similarly for $y_i(t)$,

a_i is a constant (Lanchester) attrition-rate coefficient (reflecting the effectiveness of Y_i fire against X_i),

c_i is a constant (Lanchester) attrition-rate coefficient (reflecting the effectiveness of W supporting fires against Y_i),

t_f (with numerical value T) denotes the end of the optimal control problem, and

ϕ_i denotes the fraction of W fire support directed at Y_i.

It will be convenient to consider the single control variable ϕ defined by

$$\phi = \phi_1 \quad \text{so that} \quad \phi_2 = (1-\phi) \quad \text{and} \quad 0 \le \phi \le 1. \tag{2}$$

For $T < +\infty$ it follows that $y_i(t) > 0$ for $0 \le t \le T$. Thus, the only state variable inequality constraints (SVIC's) that must be considered are $x_i \ge 0$. However, let us further assume that the attacker's infantry force levels are never reduced to zero. This assumption may be militarily justified on the grounds that X would not attack the Y_i positions if his attacking X_i forces could not survive the "approach to contact."

3.2. The Criterion Functionals Considered.

The four criterion functionals for which the optimal time-sequential fire-support allocation policies will be compared are given in Table I. All are functions only of the various numbers of combatants at the end of the planning horizon (i.e. at the end of the "approach to contact" at which time the supporting fires must be lifted for safety reasons).

The criterion functional for Problem 1 (i.e. $J_1 = \sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$) represents a weighted average of the force ratios of opposing numbers of infantry in the two infantry combat zones. The rationale behind this choice is that in each combat area (i.e. the area of combat between X_i and Y_i) combat (possibly hand-to-hand) between the X_i and Y_i forces will follow the "approach to contact" and the (initial) force ratio will be related to the outcome of this subsequent combat action. The weighting factors (i.e. α_k for $k = 1,2$) allow one to assign relative weights to this subsequent combat between X_i and Y_i in the two combat areas.

The criterion functional for Problem 2 (i.e. $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$) represents the difference between the military worths (computed using linear utilities) of the surviving X and Y forces at the end of the "approach to contact." As noted above in Section 2, we observe that maximizing the difference in worth of survivors is the same as minimizing the loss difference in combat problems (such as the one at hand)

Problem	Criterion Functional, J
1	$\sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$
2	$\sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$
3	$\left\{\sum_{k=1}^{2} v_k x_k(T)\right\}/\left\{\sum_{k=1}^{2} w_k y_k(T)\right\}$
4	$-\left\{\sum_{k=1}^{2} v_k (x_k^o - x_k(T))\right\}/\left\{\sum_{k=1}^{2} w_k (y_k^o - y_k(T))\right\}$

TABLE 1.

Summary of Problems Considered to Study Effect of Criterion Functional on Optimal Fire-Support Policy.

with no replacements.† The criterion functional for Problem 3 (i.e. $J_3 = \left\{\sum_{k=1}^{2} v_k x_k(T)\right\} / \left\{\sum_{k=1}^{2} w_k y_k(T)\right\}$) represents the ratio of total military worths of the surviving X and Y forces, whereas the one for Problem 4 (i.e. $J_4 = -\left\{\sum_{k=1}^{2} v_k (x_k^o - x_k(T))\right\} / \left\{\sum_{k=1}^{2} w_k (y_k^o - y_k(T))\right\}$) represents the ratio of military worths of losses. Both the loss ratio and the loss difference have been proposed by Pugh and Mayberry [23] as appropriate payoffs for the evaluation of combat strategies. They state that (see p. 869 of [23]) "when the most straightforward estimate of a weighting factor for the loss difference is used, the two criteria are almost equivalent." From the study at hand, we will see that a similar statement is true: the two criteria are equivalent for a certain "natural" valuation of forces (see next section), but otherwise they may yield slightly different optimal time-sequential fire-support policies.

3.3. Optimal Fire-Support Policies.

In this section we give the optimal time sequential fire-support policies for the four problems†† stated in the previous section. In all cases we assume that neither of the attacking infantry forces can be reduced to a zero force level during the approach to contact.††† Under this condition the solutions†††† to the first three problems are given in Table II with ancillary information on switching times being given in Table III. The solution to Problem 4 is exactly like that to Problem 3 except that J_3 in Problem 3 is replaced by $(-J_4)$.

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†This result also holds for problems with a fixed-length planning horizon in which rates of replacement are solely dependent on time (and not subject to control).

††As shown in Tables I and II, each of these problems corresponds to a different criterion functional for the attackers.

†††Initial force levels and the known length of the approach to contact may be sufficient to guarantee this for a given set (or range of values) of Lanchester attrition-rate coefficients.

††††For a discussion of the distinction between open-loop and closed-loop time-sequential policies, see [31] or [35]. For deterministic models such as the ones under consideration, the two types of policies are well known to be equivalent.

Table II. Optimal Fire-Support Policies for the Three Problems.†

PROBLEM 1: $$J_1 = \sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$$

<u>For $0 \le t \le T$,</u> optimal (open-loop) time-sequential fire-support policy is

$$\phi^*(t;r_1^o,r_2^o,T) = \begin{cases} 1 & \text{for } F_1(r_1^o,T) \ge F_2(r_2^o,T), \\ 0 & \text{for } F_1(r_1^o,T) \le F_2(r_2^o,T), \end{cases}$$

where

$$r_i = x_i/y_i,$$

and

$$F_i(r_i^o,T) = \alpha_i a_i c_i\left\{\left(\frac{r_i^o}{a_i}\right)\left(\frac{e^{c_iT}-1}{c_i}\right) - \frac{1}{c_i^2}\left(e^{c_iT}-1-c_iT\right)\right\}.$$

PROBLEM 2: $$J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$$

and

PROBLEM 3: $$J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right)\Big/\left(\sum_{k=1}^{2} w_k y_k(T)\right)$$

<u>Nonrestrictive Assumption:</u> $w_1/(a_1v_1) \ge w_2/(a_2v_2)$

Optimal (closed-loop) time-sequential fire-support policy is

<u>PHASE I for $0 \le t < t_1 = T - \tau_1(y_1^f/y_2^f)$</u>

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for } y_1/y_2 > a_2c_2v_2/(a_1c_1v_1), \\ c_2/(c_1+c_2) & \text{for } y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \\ 0 & \text{for } y_1/y_2 < a_2c_2v_2/(a_1c_1v_1), \end{cases}$$

<u>PHASE II for $T - \tau_1(y_1^f/y_2^f) \le t \le T$</u>

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = 1,$$

where

$$\tau_1 = \begin{cases} \tau_S & \text{for } \rho^f \ge \rho_S^f, \\ \tau_\phi & \text{for } \rho_L \le \rho^f < \rho_S^f, \\ 0 & \text{for } \rho^f < \rho_L, \end{cases}$$

$$\rho = y_1/y_2, \quad \text{and} \quad \rho_L = \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right)\Big/\left(\frac{w_1}{a_1v_1}\right).$$

NOTES:††

(1) τ_S is the unique nonnegative root of $F(\tau=\tau_S) = 0$.

(2) For $\rho_L < \rho^f < \rho_S^f$, τ_ϕ is the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$.

†It is assumed that problem parameters and initial force levels are such that $x_i(T) > 0$ for $i = 1,2$.

††See Table III for the definitions of $F(\tau)$ and $G(\tau;\rho^f)$. These functions are <u>different</u> for Problems 2 and 3.

Table III. Determination of the Switching Times τ_S and τ_ϕ for Problems 2 and 3.

<u>Nonrestrictive Assumption</u>: $w_1/(a_1v_1) \geq w_2/(a_2v_2)$

τ_S is the unique nonnegative root of $F(\tau=\tau_S) = 0$. For $\rho_L < \rho^f < \rho_S^f$, τ_ϕ is the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$.

It has been shown tha

(a) bounds on τ_ϕ are given by $0 \leq \tau_\phi < \tau_S$,

(b) τ_ϕ is a strictly increasing function of ρ^f for $\rho_L \leq \rho^f < \rho_S^f$,

(c) there is no root to $G(\tau=\tau_\phi;\rho^f) = 0$ for $\rho^f > \rho_S^f$.

<u>For PROBLEM 2</u>: $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$

$$F(\tau) = \tau + \left(\frac{1}{c_1} - \frac{w_1}{a_1v_1}\right)e^{-c_1\tau} - \left(\frac{1}{c_1} - \frac{w_2}{a_2v_2}\right)$$

$$G(\tau;\rho^f) = \frac{1}{c_1}(e^{c_1\tau}-1)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + \left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)$$

<u>Bounds on τ_S are given by</u>:

(a) For $w_1/(a_1v_1) \leq 1/c_1$,

$$\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2} \leq \tau_S \leq \frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\}.$$

(b) For $1/c_1 \leq w_1/(a_1v_1)$,

$$\frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\} \leq \tau_S \leq \frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2}\ .$$

<u>For PROBLEM 3</u>: $J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right)/\left(\sum_{k=1}^{2} w_k y_k(T)\right)$

$$F(\tau) = \tau + \left(\frac{1}{c_1} - \frac{J_3w_1}{a_1v_1}\right)e^{-c_1\tau} - \left(\frac{1}{c_1} - \frac{J_3w_2}{a_2v_2}\right)$$

$$G(\tau;\rho^f) = \frac{1}{c_1}(e^{c_1\tau}-1)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + J_3\left\{\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)\right\}$$

<u>Bounds on τ_S are given by</u>:

(a) For $J_3w_1/(a_1v_1) \leq 1/c_1$,

$$J_3\left(\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2}\right) \leq \tau_S \leq \frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\}.$$

(b) For $1/c_1 \leq J_3\, w_1/(a_1v_1)$,

$$\frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\} \leq \tau_S \leq J_3\left(\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2}\right)\ .$$

<u>Also</u>

$$\frac{\partial \tau_S}{\partial J_3} > 0.$$

Let us sketch here the proofs of a few statements made in Tables II and III. The existence of a unique nonnegative root to $F(\tau=\tau_S) = 0$ for $w_1/(a_1v_1) \geq w_2/(a_2v_2)$ follows from $F(\tau=0) \leq 0$ and $F'(\tau) > 0 \ \forall \tau \geq 0$. The existence of two positive roots to $G(\tau=\tau_\phi;\rho^f) = 0$ [here the second argument, ρ^f, is a (fixed) parameter] for $w_1/(a_1v_1) \geq w_2/(a_2v_2)$ and $\rho_L < \rho^f < \rho_S^f$ follows from $G(\tau=0) > 0$ for $\rho^f > \rho_L$ and the fact that (letting $\tilde{\tau}$ denote the unique value of τ at which the global minimum of the strictly convex function $G(\tau)$ occurs) $G(\tau=\tilde{\tau};\rho^f) = F(\tilde{\tau}) < 0$ for $\rho^f < \rho_S^f$. The latter is a consequence of $\partial G/\partial\rho^f > 0$ and $G(\tau=\tau_S;\rho^f=\rho_S^f) = F(\tau=\tau_S) = 0$. It should be noted that the fact that $G'(\tau=\tilde{\tau};\rho^f) = 0$ allows the parameter ρ^f to be eliminated from $G(\tau=\tilde{\tau};\rho^f)$. It also follows that there is no solution (i.e. value of τ_ϕ) to $G(\tau=\tau_\phi;\rho^f) = 0$ for $\rho^f > \rho_S^f$. The proof that $\partial\tau_S/\partial J_3 = -(\partial F/\partial J_3)/(\partial F/\partial\tau_S) > 0$ follows from $\partial F/\partial\tau_S > 0$ and $\partial F/\partial J_3 < 0$ (the latter holding since $\{\exp(-c_1\tau) - 1 + c_1\tau\} > 0$).

We will now illustrate the structure of the optimal time-sequential fire-support policies for the first three problems by considering some numerical examples. The basic parameter set used in the numerical computations is shown in Table IV. Numerical results have not been obtained for Problem 4 when $w_1/(a_1v_1) > w_2/(a_2v_2)$ because of the difficulty in solving the associated two-point boundary-value problem. The structure of the optimal policy, however, is similar to that for Problems 2 and 3; although switching times are, in general, very difficult to determine.

For Problem 1 it is convenient to introduce the "local" force ratio† $r_i = x_i/y_i$, which represents the ratio of the numbers of opposing infantry in each of the two combat areas (see Figure 1). The optimal time sequential fire-support policy is most conveniently expressed as an open-loop control in terms of the two initial force ratios, denoted as $r_i^o = r_i(t=0)$ for $i = 1,2$, and the given length of time for the approach to contact T. This optimal fire-support policy is graphically depicted in Figure 2. In the initial force-ratio space, the line with equation

†See [30] for some insights into the dynamics of combat from considering the force-ratio equation.

TABLE IV.

Basic Parameter Set for Numerical Examples.

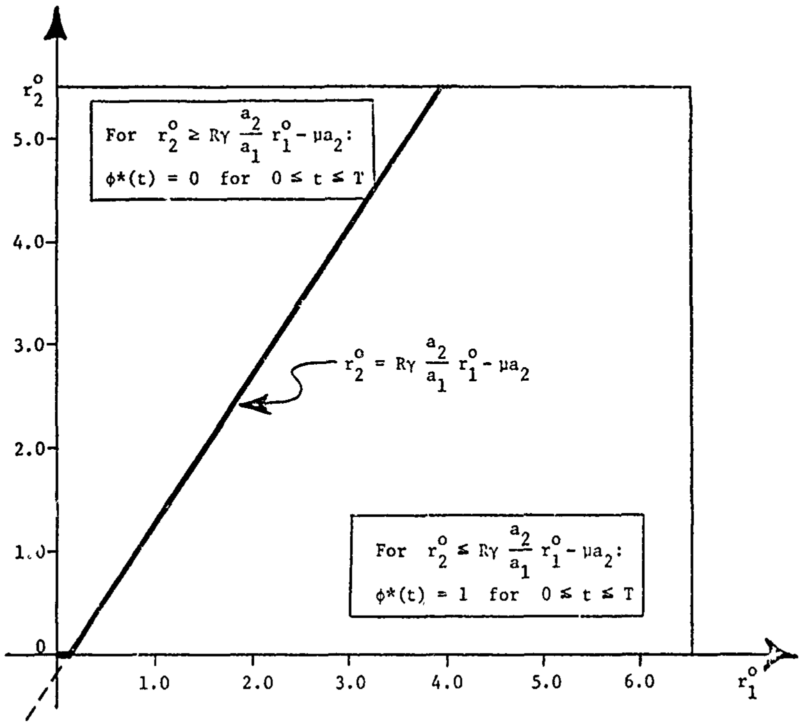
i	a_i	c_i	
1	0.020	0.06	T = 30 minutes
2	0.015	0.05	

Notes:

1. a_i has units of [X_i casualties/{(minute) × (number of Y_i)}].

2. c_i has units of [Y_i casualties/{(minute) × (number of Y_i)}].

Figure 2. Optimal (Open-Loop) Fire-Support Policy for Problem 1.



NOTES:

(1) $R = \alpha_1 a_1 c_1 / (\alpha_2 a_2 c_2)$.

(2) $\gamma = \left(\frac{c_2}{c_1}\right)\left(\frac{e^{c_1 T} - 1}{e^{c_2 T} - 1}\right)$.

(3) $\mu = \left(\frac{c_2}{e^{c_2 T} - 1}\right)\left\{\frac{R}{c_1^2}(e^{c_1 T} - 1 - c_1 T) - \frac{1}{c_2^2}(e^{c_2 T} - 1 - c_2 T)\right\}$.

(4) $r_i = x_i / y_i$.

(5) $\alpha_1 = \alpha_2 = 1.0$. See Table IV for other parameter values.

$$r_2^o = R\gamma \frac{a_2}{a_1} r_1^o - \mu a_2, \tag{3}$$

where

$$R = \alpha_1 a_1 c_1/(\alpha_2 a_2 c_2),$$

$$\gamma = \left(\frac{c_2}{c_1}\right)\left(\frac{e^{c_1T}-1}{e^{c_2T}-1}\right),$$

and

$$\mu = \left(\frac{c_2}{e^{c_2T}-1}\right)\left\{\frac{R}{c_1^2}\,(e^{c_1T}-1-c_1T) - \frac{1}{c_2^2}\left(e^{c_2T}-1-c_2T\right)\right\},$$

is a "dispersal line" (see [12], [24], or [31]) away from which all optimal battle trajectories flow. This is shown in Figure 3. In constructing this figure, we have used facts like the following: when $\phi = 1$ for $0 \leq t \leq T$ and $r_2^f = 0$, then

$$r_1 = \frac{1}{c_1}\,\{(c_1 r_1^f - a_1)e^{-c_1 r_2/a_2} + a_1\}. \tag{4}$$

For Problems 2, 3, and 4, the optimal fire-support policy (expressed as a closed-loop control (see [10] or [35])) is most conveniently expressed in terms of y_1/y_2 (i.e. the ratio of the numerical strengths of the two defending infantry forces) and $\tau = T - t$ (i.e. the "backwards" time or "time to go" in the approach to contact). When enemy forces are valued in direct proportion to the rate at which they destroy value of the friendly forces, i.e.

$$w_i = k a_i v_i \qquad \text{for } i = 1,2, \tag{5}$$

the optimal fire-support policy takes a particularly simple form (denoted as POLICY A):

POLICY A: For $0 \leq t \leq T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for } y_1/y_2 > a_2c_2v_2/(a_1c_1v_1), \\ c_2/(c_1+c_2) & \text{for } y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \\ 0 & \text{for } y_1/y_2 < a_2c_2v_2/(a_1c_1v_1). \end{cases} \tag{6}$$

NOTE: The definitions of the quantities R, γ, μ, and r_1 are given in Figure 2.

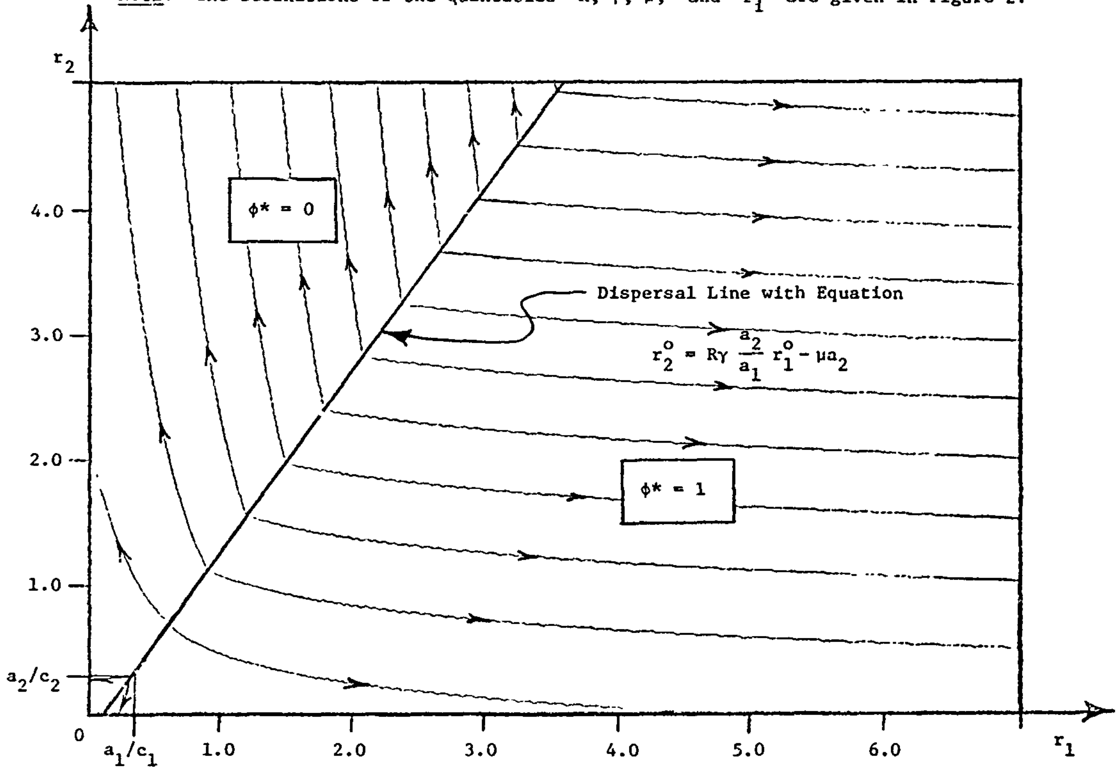


Figure 3. Optimal Battle Trajectories Resulting from Optimal (Open-Loop) Fire-Support Policy for Problem 1.

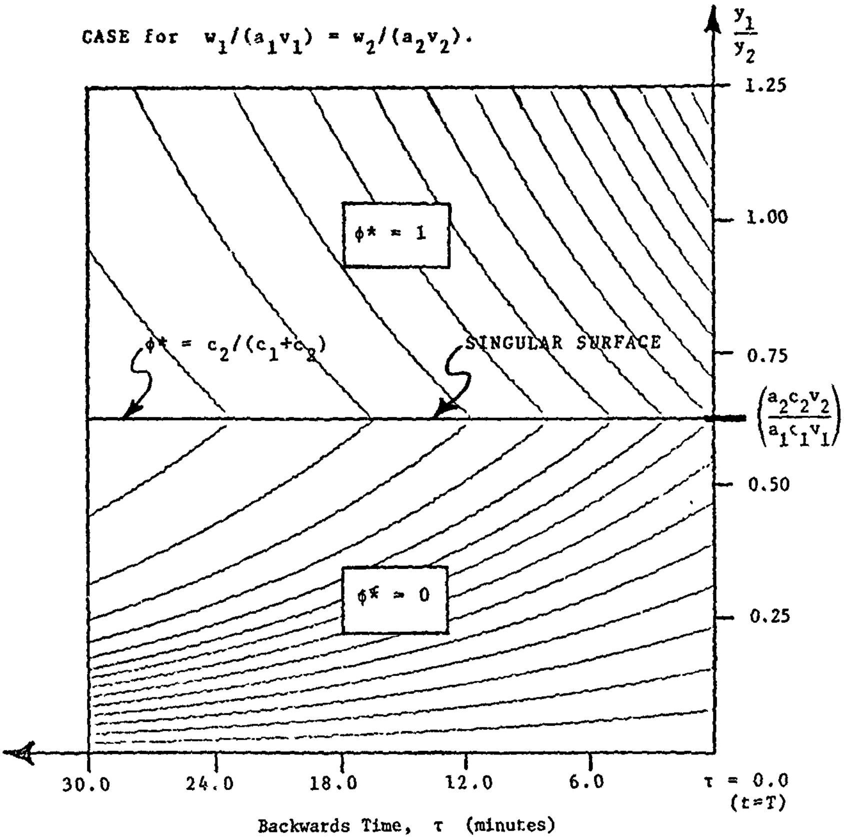
This is shown pictorially in Figure 4 in which optimal trajectories are traced backwards in time. It is convenient to note that, for example, when $\phi(\tau)$ = CONSTANT for $0 \leq \tau \leq \sigma$, we have

$$\rho(\tau) = \rho^f \exp\{[\phi c_1 - (1-\phi)c_2]\tau\}.$$

In this case, $\tau_1 = 0$ (see Table II), i.e. the entire approach to contact is "PHASE I."

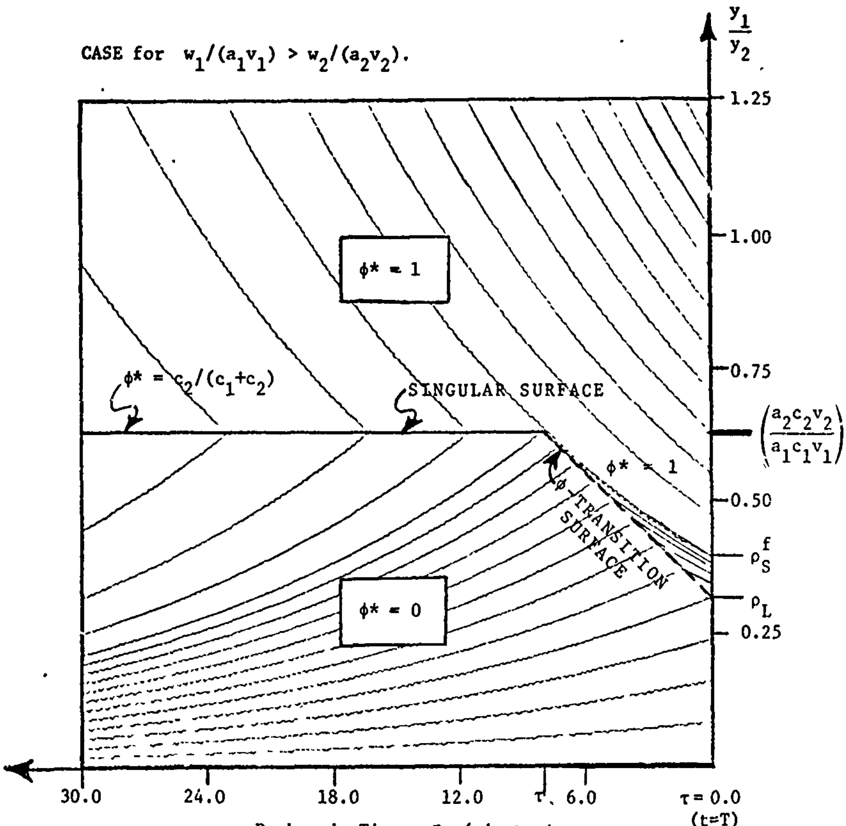
When enemy forces are not valued in direct proportion to the rate at which they destroy value of the friendly forces (without loss of generality we may assume that $w_1/(a_1v_1) > w_2/(a_2v_2)$), the solutions to Problems 2 and 3 are considerably more complex as shown in Figure 5. As we see from Table II, the planning horizon may be considered to consist of two phases (denoted as PHASE I and as PHASE II), during each of which a different fire-support allocation rule is optimal. We denote this overall optimal policy as POLICY B (see Table II). During PHASE I, POLICY A is optimal; whereas during PHASE II, it is optimal to always concentrate all artillery fire on Y_1 (which has been valued disproportionately high).

The absence or presence of PHASE II itself in the optimal time-sequential fire-support policy depends on the ratio of enemy infantry strengths $\rho = y_1/y_2$. For Problem 2 the length of PHASE II (i.e. τ_1) is <u>independent</u> of the final force levels of the attacking friendly infantry units (i.e. x_1^f and x_2^f) and depends only on $\rho^f = y_1^f/y_2^f$ and the combat effectiveness parameters (see equations (1)), whereas for Problem 3 the length of PHASE II does depend directly on x_1^f and x_2^f through the criterion functional $J_3 = \left\{\sum_{k=1}^{2} v_k x_k^f\right\}/\left\{\sum_{k=1}^{2} w_k y_k^f\right\}$. Thus, we see that τ_1 may be quite different for Problems 2 and 3: for example, for the parameter set shown in Table IV (plus force utility values $v_1 = v_2 = 15.0$, $w_1 = 4.0$, and $w_2 = 1.5$ and terminal values $x_1^f = x_2^f = 200.0$ and $y_2^f = 50.0$), we have τ_S(Problem 2) = 7.93 minutes, while τ_S(Problem 3) = 11.37 minutes. [For computing τ_S(Problem 3) by using $F(\tau)$ shown in Table III, we have used the fact that $(y_1^f)_S = y_2^f\left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\exp(-c_1\tau_S)$ to eliminate y_1^f from J_3.] Recalling that $\partial\tau_S/\partial J_3 > 0$ (see Table III above) and observing that



NOTE: See Table IV for parameter values.

Figure 4. Diagram of Optimal (Closed-Loop) Fire-Support Policy (POLICY A) for Problems 2, 3, and 4 when $w_1/(a_1v_1) = w_2/(a_2v_2)$.



NOTES: (1) $\rho = y_1/y_2$.

(2) See Table II for definitions of ρ_L and ρ_S^f.

(3) $v_1 = v_2 = 15.0$, $w_1 = 4.0$, and $w_2 = 1.5$. See Table IV for other praimeter values.

Figure 5. Diagram of Optimal (Closed-Loop) Fire-Support Policy (POLICY B) for Problem 2 when $w_1/(a_1v_1) > w_2/(a_2v_2)$. [The structure of the optimal fire-support policy is similar for Problems 3 and 4.]

$$\lim_{J_3\to+\infty} \tau_S(\text{Problem 3}) = (1/c_1)\ln\left\{\frac{[w_1/(a_1v_1)]}{[w_2/(a_2v_2)]}\right\}, \tag{7}$$

we see that for this parameter set the largest that τ_S(Problem 3) may be is $\lim_{J_3\to+\infty} \tau_S$(Problem 3) = 11.55 minutes. Thus, for this parameter set τ_S(Problem 2) and τ_S(Problem 3) may differ by at most fifty percent.

3.4. Discussion of Comparison.

In this section we will contrast the structure of the optimal time-sequential fire-support policies for the four problems considered above. Let us recall that in all cases we have assumed that $x_1^f, x_2^f > 0$.

For Problem 1 the optimal fire-support policy is to always concentrate all artillery fire (i.e. supporting fires) on just one of the two opposing enemy infantry units. This policy will maximize the force ratio at the end of the approach to contact in one of the combat areas (i.e. x_i^f/y_i^f) and may be considered to be a "break-through" tactic. In other words, one concentrates all fire support on the key enemy unit in order to overwhelm it and effect a penetration.

On the other hand, for Problems 2, 3, and 4 the optimal fire-support policy may involve splitting of fires between the two enemy troop concentrations. This property of the solution has been anticipated in Taylor's earlier work on the optimal control of "linear-law" Lanchester-type attrition processes [25], [26] (see also [34]). We may consider this policy to be an "attrition" tactic which aims to wear down the overall enemy strength. The structures of the optimal policies for Problems 2, 3, and 4 are similar, although the switching times (i.e. τ_ϕ and τ_S) may be appreciably different when enemy forces are not valued in direct proportion to the rate at which they destroy value of the friendly forces. In such a case we may assume without loss of generality that

$$w_1/(a_1v_1) > w_2/(a_2v_2). \tag{8}$$

The functional dependences of these switching times are also different in Problems 2, 3, and 4. For Problem 2 the switching times (i.e. the ϕ-transition surface) are __independent__ of the force levels of the attacking friendly forces (i.e. x_1 and x_2), as is the optimal policy itself. For Problem 3 the switching times depend (see Table III) on the ratio of military worths of surviving infantry forces (computed using linear utilities), i.e. $J_3 = \{\sum_{k=1}^{2} v_k x_k(T)\}/\{\sum_{k=1}^{2} w_k y_k(T)\}$. It has been shown (see Section 3.3 above) that $\partial\tau_S/\partial J_3 > 0$ so that the larger that J_3 becomes, the more time that is spent concentrating fire on Y_1, although there is an upper limit to this time (see (7)). Similar results hold for Problem 4, only with J_3 replaced by $(-J_4)$. For comparing the switching times between Problems 3 and 4, we note that $J_3 > (-J_4)$ if and only if $J_3 > \{\sum_{k=1}^{2} v_k x_k^o\}/\{\sum_{k=1}^{2} w_k y_k^o\}$.

The most significant thing to be noted in comparing the optimal fire-support policies for these four problems is that the entire structure of the optimal policy may be changed merely by changing the criterion functional. In particular, singular subarcs (i.e. the splitting of W's fire between Y_1 and Y_2) do not appear in the solution to Problem 1, even though the necessary conditions for optimality on singular subarcs are __exactly__ the same in all four of these problems. Such singular subarcs are, of course, part of the solution for Problems 2, 3, and 4.

4. Development of Optimal Policy for Problem 1.

The optimal policy is developed by application of modern optimal control theory. For Problem 1 it is convenient to introduce the force ratio in the $i^{\underline{th}}$ combat zone $r_i = x_i/y_i$. Then Problem 1 may be written as

$$\underset{\phi_i(t)}{\text{maximize}} \quad \sum_{k=1}^{2} \alpha_k r_k(T) \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dr_i}{dt} = -a_i + \phi_i c_i r_i \quad \text{for } i = 1,2, \tag{9}$$

$$\phi_1 + \phi_2 = 1, \quad \phi_i \geq 0, \quad \text{and} \quad r_i \geq 0 \quad \text{for} \quad i = 1,2,$$

where we recall (2). We also recall that we have assumed that $r_i > 0$.

4.1. Necessary Conditions of Optimality.

The Hamiltonian [4] is given by (using (2))

$$H = \lambda_1(-a_1+\phi c_1 r_1) + \lambda_2(-a_2+(1-\phi)c_2 r_2), \tag{10}$$

so that the maximum principle yields the extremal control law

$$\phi^*(t) = \begin{cases} 1 & \text{for } S_\phi(t) > 0, \\ 0 & \text{for } S_\phi(t) < 0, \end{cases} \tag{11}$$

where $S_\phi(t)$ denotes the ϕ-switching function defined by

$$S_\phi(t) = c_1\lambda_1 r_1 - c_2\lambda_2 r_2. \tag{12}$$

The adjoint system of equations (again using (2) for convenience) is given by (assuming that $r_i(T) > 0$)

$$\dot{\lambda}_i = -\phi_i^* c_i \lambda_i \quad \text{with} \quad \lambda_i(T) = \alpha_i \quad \text{for} \quad i = 1,2. \tag{13}$$

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = -a_1c_1\lambda_1 + a_2c_2\lambda_2, \quad \text{and} \quad \ddot{S}_\phi(t) = a_1c_1\lambda_1(c_1\phi) - a_2c_2\lambda_2(c_2(1-\phi)), \tag{14}$$

we see that on a singular subarc† we have [4], [15]

$$r_1/a_1 = r_2/a_2, \quad \text{and} \quad a_1c_1\lambda_1 = a_2c_2\lambda_2, \tag{15}$$

---

†See [26] for a further discussion.

with <u>singular control</u> given by

$$\phi_S = c_2/(c_1+c_2). \tag{16}$$

On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1c_1\lambda_1(c_1+c_2) > 0.$

4.2. <u>Synthesis of Extremals</u>.

In synthesizing extremals† by the usual backwards construction procedure (see, for example, [24] or [26]) it is convenient to introduce the "backwards" time defined by $\tau = T - t$. Rather than explicitly constructing extremals and determining domains of controllability (see [24], [31], [35]), it is more convenient to show that the return (i.e. value of the criterion functional) corresponding to certain extremals dominates that from others. For this purpose it suffices to determine all possible types of extremal policies as we will now do.

To this end, we write

$$S_\phi(\tau=0) = \alpha_2a_2c_2(Rr_1^f/a_1 - r_2^f/a_2), \tag{17}$$

where

$$R = \alpha_1a_1c_1/(\alpha_2a_2c_2). \tag{18}$$

Without loss generality we may <u>assume that $R \geq 1$</u>. Then by (14) we have

$$\mathring{S}_\phi(\tau=0) = \alpha_1a_1c_1 - \alpha_2a_2c_2 \geq 0, \tag{19}$$

where $\mathring{S}_\phi$ denotes the "backwards" time derivative $\mathring{S}_\phi = dS_\phi/d\tau$. Considering (14) we may write

$$\mathring{S}_\phi(\tau) = \alpha_2a_2c_2\{R(\lambda_1/\alpha_1) - (\lambda_2/\alpha_2)\}. \tag{20}$$

It follows that $S_\phi(\tau) > 0$ and $\phi^*(\tau) = 1$ $\forall\, \tau > 0$ when $S_\phi(\tau=0) \geq 0$ for $R > 1$ (also when $S_\phi(\tau=0) > 0$ for $R = 1$). We also have $S_\phi(\tau) < 0$ and $\phi^*(\tau) = 0$ $\forall\, \tau \geq 0$ when $S_\phi(\tau=0) < 0$ for $R = 1$.

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†By an extremal we mean a trajectory on which the necessary conditions of optimality are satisfied.

There may be a change in the sign of $S_\phi(\tau)$, however, when $S_\phi(\tau=0) < 0$ for $R > 1$. In this case $\phi^*(\tau) = 0$ for $0 \le \tau \le \tau_1$ and then

$$S_\phi(\tau) = \alpha_2 a_2 c_2 \{R r_1(\tau)/a_1 - \exp(c_2\tau) r_2(\tau)/a_2\}, \tag{21}$$

where τ_1 denotes the smallest value of τ such that $S_\phi(\tau=\tau_1) = 0$. It is clear that we must have $\mathring{S}_\phi(\tau=\tau_1) \ge 0$. If $\mathring{S}_\phi(\tau=\tau_1) > 0$, then we have a transition surface, and from (21) we find that

$$R r_1(t_1)/a_1 - \exp(c_2\tau_1) r_2(t_1)/a_2 = 0, \tag{22}$$

where $t_1 = T - \tau_1$. From (20) we find that

$$0 \le \tau_1 < (1/c_2) \ln R. \tag{23}$$

If $\mathring{S}_\phi(\tau=\tau_1) = 0$, the singular subarc may be entered, and then we have

$$\tau_1 = (1/c_2) \ln R \tag{24}$$

In this case we have

$$r_2^f = R r_1^f a_2/a_1 + F(R) a_2/c_2, \tag{25}$$

where $r_i^f = r_i(t=T)$ and $F(R) = 1 + R(\ln R - 1)$. We easily see that $F(R) > 0$ for $R > 1$. When $R = 1$, we see that once the singular subarc is entered (in forwards time), it is never exited by an extremal trajectory.

For the purposes of determining the optimal policy it suffices to consider the following four extremal policies:

$$\text{Policy 0:} \quad \phi^*(t) = 0 \quad \text{for } 0 \le t \le T, \tag{26}$$

$$\text{Policy 1:} \quad \phi^*(t) = 1 \quad \text{for } 0 \le t \le T, \tag{27}$$

$$\text{Policy B-B:} \quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t < T-\tau_1, \\ 0 & \text{for } T-\tau_1 \le t \le T, \end{cases} \tag{28}$$

where $0 \le \tau_1 < (1/c_2)\ln R$, and†

$$\text{Policy S:} \quad \phi^*(t) = \begin{cases} c_2/(c_1+c_2) & \text{for } 0 \le t < T-\tau_1, \\ 0 & \text{for } T-\tau_1 \le t \le T, \end{cases} \tag{29}$$

† The only extremal policies that are omitted here are those corresponding to extremals which contain a singular subarc but $r_1^o/a \ne r_2^o/a$.

where $\tau_1 = (1/c_2)\ln R$ and $r_1^o/a_1 = r_2^o/a_2$. It is readily seen from (17) that Policy 0 yields $Rr_1^f/a_1 \geq r_2^f/a_2$, etc. We also note that corresponding to the bang-bang policy (28) we have

$$\begin{aligned} r_1(t_1) &= \{(c_1 r_1^o - a_1)\exp(c_1 t_1) + a_1\}/c_1, \\ r_2(t_1) &= r_2^o - a_2 t_1 \geq 0. \end{aligned} \tag{30}$$

4.3. <u>Determination of the Optimal Fire-Support Policy</u>.

As we have discussed elsewhere [25]-[27], [31], [35], the optimality of an extremal trajectory may be proven via citing the appropriate existence theorem for an optimal control; for the problem at hand there are two further subcases: (1) if the extremal is unique, then it is optimal, or (2) if the extremal is not unique and only a finite number exist, then the optimal trajectory is determined by considering the finite number of corresponding values of the criterion functional.† The existence of a <u>measurable</u> optimal control follows by Corollary 2 on p. 262 of [17]. In Sections 4.1 and 4.2 above, we have considered necessary conditions of optimality for piecewise continuous controls (see p. 10 and pp. 20-21 of [22]). It remains to show that the measurable optimal control may be taken to be piecewise continuous. This assertion may be proven by observing that if we consider the maximum principle for measurable controls†† (see p. 81 of [22]) in the backwards synthesis of extremals, then the optimal control may be taken to be piecewise constant (and hence piecewise continuous).†††

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†It has not been possible to establish the optimality of a policy by citing one of the many sets of sufficient conditions that are available (see [4], [26], [35]). In particular, although the planning horizon for the problem at hand is of fixed length, one cannot invoke the sufficient conditions based on convexity of Mangasarian [18] or Funk and Gilbert [6] because the right-hand sides of the differential equations (9) are not concave functions of r_i and ϕ_i.

††We have taken the liberty of changing the sign of the adjoint vector of Pontryagin et al. [22] (see p. 108 of [4]). When the admissable controls are measurable and bounded, the Hamiltonian (10) only attains its maximum almost everywhere in time.

†††This assertion follows from the control variable appearing linearly in the Hamiltonian (10), the control variable space being compact, and the switching function (12) being continuous for $0 \leq t \leq T$. The maximum principle (also singular control considerations) then yields that the optimal control must be piecewise constant almost everywhere, since

(cont. next page)

We will now show that the optimal control must be constant.† This is done by showing that the returns from both Policy B-B and also Policy S†† for a given point in the initial state space are dominated by the return corresponding to a constant extremal control. We denote the value of the criterion functional corresponding to Policy 0 as J_0, that corresponding to Policy B-B as J_B, etc. Then we have

$$J_0 = \alpha_2 a_2 c_2 \left\{ \left(\frac{\mathring{r}_1}{a_1}\right)\frac{R}{c_1} - \left(\frac{\mathring{r}_2}{a_2}\right)\frac{1}{c_2}\exp(c_2T) - \left[\frac{RT}{c_1} + \frac{1}{c_2^2}\left(\exp(c_2T)-1\right)\right]\right\}, \tag{31}$$

$$J_1 = \alpha_2 a_2 c_2 \left\{ \left(\frac{\mathring{r}_1}{a_1}\right)\frac{R}{c_1}\exp(c_1T) + \left(\frac{\mathring{r}_2}{a_2}\right)\frac{1}{c_2} - \left[\frac{R}{c_1^2}\left(\exp(c_1T)-1\right) + \frac{T}{c_2}\right]\right\}. \tag{32}$$

$$\begin{aligned} J_B = \alpha_2 a_2 c_2 \Bigg\{ & \left(\frac{\mathring{r}_1}{a_1}\right)\frac{R}{c_1}\exp(c_1[T-\tau_1]) + \left(\frac{\mathring{r}_2}{a_2}\right)\frac{1}{c_2}\exp(c_2\tau_1) \\ & - \frac{R}{c_1^2}\left[\exp(c_1[T-\tau_1]) - 1 + c_1\tau_1\right] - \frac{1}{c_2^2}\left[(1+c_2[T-\tau_1])\exp(c_2\tau_1) - 1\right]\Bigg\}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} J_S = \alpha_2 a_2 c_2 \Bigg\{ & \left(\frac{\mathring{r}_1}{a_1}\right)\frac{R^{\alpha}}{K}\exp(KT) - \Bigg[\frac{R}{K^2}\left(R^{-\beta}\exp(KT)-1\right) \\ & + \frac{1}{c_1c_2}R\ln R + \frac{1}{c_2^2}(R-1)\Bigg]\Bigg\}, \end{aligned} \tag{34}$$

where $\alpha = c_2/(c_1+c_2)$, $\alpha+\beta = 1$, and $K = c_1c_2/(c_1+c_2)$. It is convenient to define $\Delta J_{1-0} = J_1 - J_0$, etc., and then

$$\begin{aligned} \Delta J_{1-0} = \alpha_2 a_2 c_2 \Bigg\{ & R\left[\left(\frac{\mathring{r}_1}{a_1}\right)\left(\frac{\exp(c_1T)-1}{c_1}\right) - \frac{1}{c_1^2}\left(\exp(c_1T)-1-c_1T\right)\right] \\ & - \left[\left(\frac{\mathring{r}_2}{a_2}\right)\left(\frac{\exp(c_2T)-1}{c_2}\right) - \frac{1}{c_2^2}\left(\exp(c_2T)-1-c_2T\right)\right]\Bigg\}, \end{aligned} \tag{35}$$

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†This was first conjectured by Professor Frank Faulkner, to whom the authors express their thanks.

††By the principle of optimality (see [4]) it suffices for the purpose of showing that a singular solution is always nonoptimal to consider a singular extremal which begins with a singular subarc.

(cont. from page 19) $S_2(t)$ can change sign at most once. Hence, it may be considered to be piecewise constant (see p. 130 of [22]). [The authors wish to thank J. Wingate of Naval Surface Weapons Center, White Oak for generously pointing out this type of argument.]

$$\Delta J_{1-B} = \alpha_2 a_2 c_2 \left\{ R \left[\left(\frac{r_1^o}{a_1}\right) \left(\frac{\exp(c_1 T) - \exp(c_1[T-\tau_1])}{c_1}\right) - \frac{1}{c_1^2} (\exp(c_1 T) - \exp(c_1[T-\tau_1]) - c_1\tau_1) \right] \right.$$

$$\left. - \left[\left(\frac{r_2^o}{a_2}\right) \left(\frac{\exp(c_2\tau_1)-1}{c_2}\right) - \frac{1}{c_2^2} (\exp(c_2\tau_1) + c_2[T-\tau_1]\exp(c_2\tau_1) - 1 - c_2 T) \right] \right\}, \quad (36)$$

and†

$$\Delta J_{1-S} = \alpha_2 a_2 c_2 \left\{ \left(\frac{r_1^o}{a_1}\right) \left[\frac{1}{c_1} (R \exp(c_1 T) - 1) - \frac{1}{K} (R^{\alpha} \exp(KT) - 1) \right] \right.$$

$$\left. + \frac{R}{K^2} \left(R^{-\beta} \exp(KT) - 1 - \frac{KT}{R}\right) - \frac{R}{c_1^2} \left(\exp(c_1 T) - 1 - \frac{c_1 T}{R}\right) + \frac{1}{c_1 c_2} R \ln R + \frac{1}{c_2^2} (R-1) \right\} . \quad (37)$$

We now state and prove Lemma 1.

> LEMMA 1: Assume that $T \geq \tau_1$. If $\Delta J_{1-0} \geq 0$, then $\Delta J_{1-B} \geq 0$.

PROOF: (a) We consider for $t \geq \tau_1$

$$F(t) = R \left\{ \left(\frac{r_1^o}{a_1}\right) \left(\frac{\exp(c_1 t) - \exp(c_1[t-\tau_1])}{c_1}\right) - \frac{1}{c_1^2} (\exp(c_1 t) - \exp(c_1[t-\tau_1]) - c_1\tau_1) \right\}$$

$$- \left\{ \left(\frac{r_2^o}{a_2}\right) \left(\frac{\exp(c_2\tau_1)-1}{c_2}\right) - \frac{1}{c_2^2} (\exp(c_2\tau_1) + c_2[t-\tau_1]\exp(c_2\tau_1) - 1 - c_2 t) \right\} .$$

Then $\Delta J_{1-0} \geq 0 \Leftrightarrow F(t=\tau_1) \geq 0$.

(b) We compute that

$$F'(t) = R\{\exp(c_1 t) - \exp(c_1[t-\tau_1])\} \left\{ \left(\frac{r_1^o}{a_1}\right) - \frac{1}{c_1} (1 - \exp(-c_1\tau_1)) \right\} + \frac{1}{c_2} (\exp(c_2\tau_1) - 1).$$

(c) If $c_1 r_1^o \leq a_1$, then $dr_1/dt(t) \leq 0$ for $0 \leq t \leq t_1$ so that $(r_1^o/a_1) \geq (r_1(t_1)/a_1) \geq \tau_1$. It follows that $F'(t) \geq 0$. If $c_1 r_1^o > a_1$, then $F'(t) > 0$. Thus, we always have $F'(t) \geq 0$ for $t \geq \tau_1$.

(d) By (a) and (c), we have $F(t) \geq 0$, whence follows the lemma. Q.E.D.

> LEMMA 2: For $t_1 = T - \tau_1 \geq 0$, we have $\Delta J_{0-B} \geq 0$ with $\Delta J_{0-B} > 0$ for $t_1 > 0$.

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†In computing ΔJ_{1-S} we assume that $r_1^o/a_1 = r_2^o/a_2$.

PROOF: (a) We consider for $t_1 \geq 0$

$$F(t_1) = -R\left\{\left(\frac{r_1^o}{a_1}\right)\left(\frac{\exp(c_2 t_1)-1}{c_2}\right) - \frac{1}{c_1^2}\left(\exp(c_1 t_1) - 1 - c_1 t_1\right)\right\}$$

$$+ \exp(c_2 \tau_1)\left\{\left(\frac{r_2^o}{a_2}\right)\left(\frac{\exp(c_2 t_1)-1}{c_2}\right) - \frac{1}{c_2^2}\left(\exp(c_2 t_1) - 1 - c_2 t_1\right)\right\}.$$

We observe that $F(t_1=0) = 0$.

(b) We compute that $F'(t_1) = -\frac{R}{a_1}\left\{\frac{1}{c_1}[(c_1 r_1^o - a_1)\exp(c_1 t_1) + a_1]\right\} + \frac{\exp(c_2\tau_1)}{a_2} \cdot$ $\{r_2^o \exp(c_2 t_1) - \frac{a_2}{c_2}(\exp(c_2 t_1) - 1)\}$. Considering (22) and (30), we find that for $t_1 \geq 0$ we have

$$F'(t_1) = \exp(c_2\tau_1)\left\{\left(\frac{r_2^o}{a_2}\right)(\exp(c_2 t_1)-1) - \frac{1}{c_2}(\exp(c_2 t_1) - 1 - c_2 t_1)\right\}.$$

(c) Recalling (30) that $r_2^o/a_2 \geq t_1$, we have for $t_1 \geq 0$

$$F'(t_1) \geq \exp(c_2\tau_1)\{t_1(\exp(c_2 t_1)-1) - \frac{1}{c_2}(\exp(c_2 t_1) - 1 - c_2 t_1)\} \geq 0,$$

since for $t \geq 0$ we have $g(t) \geq 0$, where $g(t) = t(\exp(c_2 t)-1) - (\exp(c_2 t)-1-c_2 t)/c_2$. The latter result follows from $g(t=0) = 0$ and $g'(t) \geq 0 \; \forall \; t \geq 0$.

(d) Thus, $F(t_1) \geq 0 \; \forall \; t_1 \geq 0$, whence follows the lemma. Q.E.D.

As an immediate consequence of Lemmas 1 and 2 we have Theorem 1.

THEOREM 1: For $T \geq \tau_1 > 0$, we have $\max(J_0, J_1) \geq J_B$ with strict inequality holding for $T > \tau_1$.

We next consider Lemma 3.

LEMMA 3: Assume that $R \geq 1$ and $T \geq \tau_1$. Then we have $\Delta J_{1-S} \geq 0$ with $\Delta J_{1-S} > 0$ for $R > 1$.

PROOF: (a) We consider for $t \geq 0$

$$F(t) = t\{(R\exp(c_1 t) - 1)/c_1 - (R^{\alpha}\exp Kt - 1)/K\} + R(R^{-\beta}\exp(KT) - 1 - Kt/R)/K^2$$
$$- R(\exp(c_1 t) - 1 - c_1 T/R)/c_1^2 + (R \ln R)/(c_1 c_1) + (R-1)/c_2^2.$$

Then we have

$$F(t=0) = R(R^{-\beta}-1)/K^2 + (R \ln R)/(c_1 c_2) + (R-1)/c_2^2 = f(R) \geq 0,$$

with $f(R) > 0$ for $R > 1$. The latter result follows from $f(R=1) = f'(R=1) = 0$ and $f''(R) = (1-R^{-\beta})/(c_1 c_2 R) > 0 \ \forall \ R > 1$.

(b) Computing $F'(t) = R^{\alpha}t\{R^{\beta}\exp(c_1 t) - \exp(Kt)\} \geq R^{\alpha}t\{\exp(c_1 t) - \exp(Kt)\} > 0$ for $R \geq 1$ and $t > 0$, we see from (a) that $F(t;R) \geq 0$ with $F(t;R) > 0$ for $R > 1$.

(c) We now consider $G(t) = \{R\exp(c_1 t) - 1\}/c_1 - \{R^{\alpha}\exp(Kt) - 1\}/K$. It follows that $G(t=0) = 1/c_2 + R/c_1 - R^{\alpha}/K = g(R) \geq 0$, since $g(R=1) = 0$ and $g'(R) = (1-R^{-\beta})/c_1$. Also $G'(t) = R^{\alpha}\{R^{\beta}\exp(c_1 t) - \exp(Kt)\} \geq 0$. Hence, $G(t) \geq 0$.

(d) Recalling that $r_1^o/a_1 \geq T$, we have by (c) that $\Delta J_{1-S} \geq \alpha_2 a_2 c_2 F(T;R) \geq 0$ with $F(T;R) > 0$ for $R > 1$. Q.E.D.

From Lemma 3 follows Theorem 2.

> THEOREM 2: Assume that $R \geq 1$ and $T \geq \tau_1$. Then $\max(J_0, J_1) \geq J_S$ with inequality holding for $R > 1$.

Thus, we see from Theorems 1 and 2 that the optimal control must be constant and equal to either 0 or 1 for $0 \leq t \leq T$. The results shown in Table II and Figures 2 and 3 then follow from consideration of ΔJ_{1-0} (see equation (35)).

5. Development of Optimal Policy for Problem 2.

In this case we consider (1) with the criterion function $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$. Thus, for this problem the state space (considering time to be an additional state variable) if five dimensional.

5.1. Necessary Conditions of Optimality.

The Hamiltonian [2] is given by (using (2))

$$H = -\sum_{i=1}^{2} p_i a_i y - q_1 \phi c_1 y_1 - q_2 (1-\phi) c_2 y_2, \tag{38}$$

so that the maximum principle yields the extremal control law

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad S_\phi(t) > 0, \\ 0 & \text{for} \quad S_\phi(t) < 0, \end{cases} \tag{39}$$

where $S_\phi(t)$ denotes the ϕ-switching function defined by

$$S_\phi(t) = c_1(-q_1)y_1 - c_2(-q_2)y_2. \tag{40}$$

The adjoint system of equations (again using (2) for convenience) is given by (assuming that $x_i(T) > 0$)

$$p_i(t) = v_i \quad \text{for} \quad 0 \le t \le T \ \text{ with } \ i = 1,2,$$

and (41)

$$\dot{q}_i = a_i v_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i \quad \text{for} \ \ i = 1,2.$$

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = -a_1 c_1 v_1 y_1 + a_2 c_2 v_2 y_2, \ \text{ and } \ \ddot{S}_\phi(t) = a_1 c_1 v_1 y_1 (c_1 \phi) - a_2 c_2 v_2 y_2 (c_2(1-\phi)), \tag{42}$$

we see that on a singular subarc we have [4], [15]

$$y_1/y_2 = a_2 c_2 v_2/(a_1 c_1 v_1), \qquad \text{and} \qquad (-q_1)/(a_1 v_1) = (-q_2)/(a_2 v_2), \tag{43}$$

with the singular control given by

$$\phi_S = c_2/(c_1 + c_2). \tag{44}$$

On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1 c_1 v_1 y_1 (c_1 + c_2) > 0.$

For Problem 1 it was convenient to consider a "reduced" state space consisting of $t, r_1 = x_1/y_1$, and r_2, while for Problem 2 we are considering the "full" state space of t, x_1, x_2, y_1, and y_2. It seems appropriate to point out the corresponding relation between the adjoint variables in these two state spaces. This relation is easily seen by considering the optimal return function (see [4]), denoted as W, and the following transformation of variables

$$t = t, \quad \text{and} \quad r_i = x_i/y_i \quad \text{for} \quad i = 1,2. \tag{45}$$

Then we have, for example, $p_i(t) = \frac{\partial W}{\partial x_i(t)} = \frac{\partial W}{\partial r_i}\frac{\partial r_i}{\partial x_i}$ so that we obtain

$$p_i = \lambda_i/y_i, \quad \text{and} \quad q_i = -r_i\lambda_i/y_i \quad \text{for} \quad i = 1,2. \tag{46}$$

Let us also note that, alternatively, Problem 1 could have been solved in the "full" state space of t, x_1, x_2, y_1, and y_2; while Problem 2 cannot be solved in the "reduced" state space. The latter conclusion follows from considering (41) and the requirement [see (46) above] that $p_i/q_i = -1/r_i$ must hold for the transformation (45) to be applicable.

5.2. Synthesis of Extremals.

In synthesizing extremals by the usual backwards construction procedure it is convenient to consider

$$S_\phi(\tau=0) = a_2c_2v_2y_2^f\left(\frac{w_1}{a_1v_1}\right)\left\{\frac{a_1c_1v_1y_1^f}{a_2c_2v_2y_2^f} - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\}, \tag{47}$$

and

$$\overset{\circ}{S}_\phi(\tau) = a_1c_1v_1y_1 - a_2c_2v_2y_2, \tag{48}$$

where τ denotes the "backwards" time defined by $\tau = T - t$, and $\overset{\circ}{S}_\phi$ denotes the "backwards" time derivative $\overset{\circ}{S}_\phi = dS_\phi/d\tau$. We omit most of the tedious details of the synthesis of extremals because of similarity to those in [26]. Without loss of generality we may assume that (8) holds, and then there are two cases to be considered: (I) $w_1/(a_1v_1) = w_2/(a_2v_2)$, and (II) $w_1/(a_1v_1) > w_2/(a_2v_2)$.

CASE I: $w_1/(a_1c_1) = w_2/(a_2v_2)$; i.e. $w_i = ka_iv_i$ for $i = 1,2$.

In this case (46) becomes

$$S_\phi(\tau=0) = a_2c_2v_2y_2^f(w_1/(a_1v_1))\{a_1c_1v_1y_1^f/(a_2c_2v_2t_2^f) - 1\},$$

whence follows the synthesis of extremals shown in Figure 4.

CASE II: $w_1/(a_1v_1) > w_2/(a_2v_2)$.

In this case it follows from (39), (47), and (48) that for $\rho^f = y_1^f/y_2^f \geq$

$a_2c_2v_2/(a_1c_1v_1)$ we have $S_\phi(\tau) > 0$ and $\phi^*(\tau) = 1$ for all $\tau > 0$. Since $S_\phi(\tau{=}0) \le 0 \Rightarrow \mathring{S}_\phi(\tau{=}0) < 0$, it follows that for $\rho^f \le \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)$ we have $S_\phi(\tau) < 0$ and $\phi^*(\tau) = 0$ for all $\tau > 0$.

There may be a change in the sign of $S_\phi(\tau)$, however, for $c_2w_2/(c_1w_1) < \rho^f < a_2c_2v_2/(a_1c_1v_1)$. In this case $\phi^*(\tau) = 1$ for $0 \le \tau \le \tau_1$ and then

$$S_\phi(\tau) = a_2c_2v_2y_2^f\left\{\frac{1}{c_1}[\exp(c_1\tau)-1]\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + \left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)\right\}. \quad (49)$$

It is clear that we must have $S_\phi(\tau{=}\tau_1) \le 0$. If $S_\phi(\tau{=}\tau_1) < 0$, then we have a <u>transition surface</u> with τ_1 (denoted as τ_ϕ) given by the smaller of the two positive roots of $G(\tau{=}\tau_\phi;\rho^f) = 0$, where $G(\tau;\rho^f)$ is given in Table III. If $\mathring{S}_\phi(\tau{=}\tau_1) = 0$, a <u>singular subarc</u> may be entered, and then we have that τ_1 (denoted as τ_S) is given by the unique nonnegative root of $F(\tau{=}\tau_S) = 0$, where $F(\tau)$ is given in Table III. We denote the corresponding value of ρ^f as ρ^f_S. Then there is no switch in ϕ^* for $\rho^f > \rho^f_S$. We state this result as Theorem 3.

THEOREM 3: $\phi^*(\tau) = 1$ for all $\tau \ge 0$ when $\rho^f > \rho^f_S$.

PROOF: Immediate by $G(\tau{=}\tau_S;\rho^f{=}\rho^f_S) = F(\tau{=}\tau_S) = 0$ and $\partial G/\partial\rho^f > 0$, since then there is no solution to $G(\tau{=}\tau_1;\rho^f) = 0$ for $\rho^f > \rho^f_S$. <u>Q.E.D.</u>

The bounds on τ_S shown in Table III are developed as follows. First assume that $w_1/(a_1v_1) \le 1/c_1$ and consider $F(\tau) = \tau + \{1/c_1 - w_1/(a_1v_1)\}\exp(-c_1\tau) - \{1/c_1 - w_2/(a_2v_2)\}$. Then $c_1w_1/(a_1v_1) \le F'(\tau) \le 1$ and $F''(\tau) \ge 0$ for $w_1/(a_1v_1) \le 1/c_1$, whence follow the bounds shown in Table III. Other developments are similar.

The above information immediately leads to the extremal field shown in Figure 5 (see also Tables II and III).

5.3. <u>Determination of the Optimal Fire-Support Policy.</u>

The optimality of the extremal fire-support policy developed above follows according to the reasoning given in Section 4.3 by the uniqueness of extremals.

6. Development of Optimal Policy for Problem 3.

In this case we consider (1) with the criterion functional $J_3 = \left\{\sum_{k=1}^{2} v_k x_k(T)\right\} / \left\{\sum_{k=1}^{2} w_k y_k(T)\right\}$.

6.1. Necessary Conditions of Optimality.

The necessary conditions of optimality for Problem 3 are the same as those for Problem 2 except that the boundary conditions for the adjoint variables are different. Thus, (38) through (40) also apply to Problem 3. The adjoint system of equations (again using (2) for convenience) is given by (assuming that $x_i(T) > 0$)

$$p_i(t) = v_i/D \quad \text{for} \quad 0 \le t \le T \quad \text{with} \quad i = 1,2,$$

and (50)

$$\dot{q}_i = a_i p_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i J_3/D \quad \text{for} \quad i = 1,2,$$

where $D = \sum_{k=1}^{2} w_k y_k(T)$.

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = -a_1c_1p_1y_1 + a_2c_2p_2y_2, \quad \text{and} \quad \ddot{S}_\phi(t) = a_1c_1p_1y_1(c_1\phi) - a_2c_2p_2y_2(c_2(1-\phi)), \quad (51)$$

we find that (43) and (44) again hold on a <u>singular subarc</u>. On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1c_1v_1y_1(c_1+c_2)/D > 0$.

6.2. Synthesis of Extremals.

The synthesis of extremals is essentially the same as for Problem 2 (see Section 5.2 above) except that we have

$$S_\phi(\tau=0) = J_3 a_2c_2v_2y_2^f\left(\frac{w_1}{a_1v_1}\right)\left\{\left(\frac{a_1c_1v_1y_1^f}{a_2c_2v_2y_2^f}\right) - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\}/D, \quad (52)$$

and

$$\overset{\circ}{S}_\phi(\tau) = (a_1c_1v_1y_1 - a_2c_2v_2y_2)/D. \quad (53)$$

It follows that

$$S_\phi(\tau) = \{J_3a_2c_2v_2y_2^f\left(\frac{w_1}{a_1v_1}\right)\left[\left(\frac{a_1c_1v_1y_1^f}{a_2c_2v_2y_2^f}\right) - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right] + \int_0^\tau [a_1c_1v_1y_1(\sigma) - a_2c_2v_2y_2(\sigma)]d\sigma\}/D. \quad (54)$$

6.3. Determination of the Optimal Fire-Support Policy.

As for Problem 2, the optimality of the extremal fire-support policy developed above follows according to the reasoning given in Section 4.3 by the uniqueness of extremals.

7. Development of Optimal Policy for Problem 4.

In this case we consider (1) with the criterion functional $J_4 = -\sum_{k=1}^{2} v_k[x_k - x_k(T)] / \sum_{k=1}^{2} w_k[y_k^o - y_k(T)]$. The necessary conditions of optimality for Problem 4 are the same as those for Problems 2 and 3, except that the boundary conditions for the adjoint variables are different: at $t = T$ we have

$$p_i(T) = v_i/D_1 \quad \text{and} \quad q_i(T) = -w_i(-J_4)/D_1 \quad \text{for} \quad i = 1,2, \tag{55}$$

where $D_1 = \sum_{k=1}^{2} w_k[y_k^o - y_k(T)]$. Consequently, the solution to Problem 4 is exactly the same as that to Problem 3, except that J_3 in the solution to Problem 3 is replaced by $(-J_4)$. Because of the dependence of J_4 on the initial force levels x_i^o, y_i^o for $i = 1,2$, the two-point boundary-value problem which arises in the determination of switching times when (8) holds is very difficult to solve.

8. Discussion.

In this section we discuss what we have learned about the dependence of the structure of optimal time-sequential fire-support policies on the quantification of military objectives. We studied this dependence by considering four specific problems (each corresponding to a different quantification of objectives, i.e. criterion functional) for which solutions were developed by modern optimal control theory.

Our most significant finding is that essentially the entire structure of the optimal time-sequential fire-support policy may be changed by modifying the quantification of military objectives. We feel that there are basically two types of military strategies: (1) obtain a "local" advantage, and (2) obtain an "overall" advantage. The criterion functional for Problem 1 (i.e. $J_1 = \sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$, a weighting of the final force ratios in the two separate combat areas) reflects the striving to attain a "local" advantage (referred to above as a "breakthrough" tactic). The corresponding optimal fire-support policy was to concentrate all supporting fires on one of the enemy units (the quantitative determination of this policy is given in Table II) for the entire period of fire support.†

On the other hand, the criterion functionals for Problems 2, 3, and 4†† reflect the striving to attain an "overall" advantage (referred to above as an "attrition" tactic which aims to wear down the overall enemy strength). The corresponding optimal time-sequential fire-support policies for Problems 2, 3, and 4 were qualitatively the same and could involve a splitting of supporting fires between the two enemy troop concentrations. This property of the optimal fire-distribution policy is not present in the solution to Problem 1 and was anticipated by our earlier work on optimal fire distribution against enemy target types which undergo attrition according to a "linear-law" process (see Section 3.1 above) [25], [26]. The criterion functional for this earlier work was the difference between the overall military worths of friendly and enemy survivors. Thus, we see that nonconcentration of fires on particular target types is characteristic of optimal time-sequential fire distribution over enemy target types which undergo attrition according to a "linear-law" process with the objective of attaining an "overall" advantage.

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†We have assumed that the X commander has perfect information about the state variables (e.g. enemy force levels) and all Lanchester attrition-rate coefficients (i.e. system parameters). In the real world where this assumption may not hold, this policy need not be optimal. Other factors that would temper the use of such a policy in the real world are (1) the need to "pin down" enemy forces with supporting fires (i.e. suppressive effects), and (2) the giving of information to the enemy as to exactly where his defenses will be attacked by the concentration of preparatory fires only there.

††We recall that $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$, the difference between overall military worths (computed assuming linear utilities) of friendly and enemy forces at the time when supporting fires must be lifted; $J_3 = \{\sum_{k=1}^{2} v_k x_k(T)\}/\{\sum_{k=1}^{2} w_k y_k(T)\}$, the ratio of overall military worths; and $J_4 = -\{\sum_{k=1}^{2} v_k [x_k^0 - x_k(T)]\}/\{\sum_{k=1}^{2} w_k [y_k^0 - y_k(T)]\}$, the ratio of the military worths of friendly and enemy losses.

We saw that the structures of the optimal time-sequential fire-support policies for Problems 2, 3, and 4 were qualitatively similar. In fact, when one (i.e. the X commander) values enemy (i.e. Y) forces in each of the two combat zones in direct proportion to their rate (per unit of individual weapon system) of destroying the value of opposing friendly forces, the optimal policies were exactly the same for all three problems (see Table II). In this case the optimal fire-support policy took the particularly simple form of Policy A as given by (6).

When enemy survivors were not valued in direct proportion to their rate of destruction of friendly value, the optimal policies were different and more complex (see Tables II and III; also Figure 5), and the planning horizon may be considered to be divided into two phases, denoted as PHASE I and PHASE II. The lengths of these two phases depended on different factors in these three problems, and the timing of changes in the allocation of supporting fires could be appreciably different. When the planning objective was the maximization of the difference in the total military worths of friendly and enemy forces at the end of the "approach to contact," the length of, for example, PHASE II (during which all fire is concentrated on Y_1) depended only on the attrition-rate coefficients and enemy force levels and was independent of the friendly attacking-force levels. When the ratio of the total worths of surviving friendly and enemy forces was considered (i.e. for Problem 3), the length of PHASE II also depended directly on the attacking friendly force levels; while when the ratio of the total worths of friendly and enemy losses was considered, it also depended on the initial total worths of forces.

Thus, we see that (at least for the relatively simple fire-support allocation problem considered here) the structure of the optimal time-sequential allocation policy may be strongly influenced by the quantification of military objectives. Moreover, the most important planning decision apparently is whether a side will seek to attain an "overall" advantage or a "local" advantage. We hope that our investigation has provided a better understanding of the dependence of the structure of optimal time-sequential fire-support strategies on combatant objectives. In conclusion, it appears to us that more such specific cases warrant investigation for developing a theory of optimal combat strategies.

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